JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **135**, 61–74 (1988)

Jordan Models and $*$-Cyclic Sets

HARI BERCOVICI*

Department of Mathematics, Indiana University, Bloomington, Indiana 47405

Submitted by C. Foias

Received January 27, 1987

1. INTRODUCTION

Let $\mathscr{F}$ be a separable Hilbert space, and denote by $H^2(\mathscr{F})$ the usual Hardy space consisting of all power series

$$f(\lambda)=\sum_{n=0}^{\infty} \lambda^n u_n, \qquad |\lambda|<1,$$

where $u_n \in \mathscr{F}$, and

$$\|f\|^2=\sum_{n=0}^{\infty} \|u_n\|^2<\infty.$$

Denote by U_+ the unilateral shift on $H^2(\mathscr{F})$, i.e.,

$$(U_+ f)(\lambda)=\lambda f(\lambda), \qquad \lambda \in \mathbb{D},\ f \in H^2(\mathscr{F}).$$

For a given subset $\mathscr{M} \subset H^2(\mathscr{F})$ we can form the subspace

$$\mathscr{N}=\bigvee_{n=0}^{\infty} U_+^{*n}\mathscr{M} \subset H^2(\mathscr{F}),$$

invariant for U_+^*. Our problem is as follows. Suppose that the compression $T=P_{\mathscr{N}} U_+ \mid \mathscr{N}$ is an operator of class C_0, as defined in Chapter III of [8]. We want to calculate the Jordan model of the operator T directly from its $*$-cyclic set $\mathscr{M}$, without calculating the characteristic function of T.

This problem was first considered by Frazho in [4], where the space $\mathscr{F}$ was assumed to be finite-dimensional. We generalize Frazho's results to the

* The research in this paper was supported in part by a grant from the National Science Foundation.

0022-247X/88 $3.00

409/135/1-5

general case and provide the geometric structure for understanding them. We also provide a generalization to the infinite-dimensional case of Frazho's result from [5]. We note that Frazho's proofs use the diagonalization theory of Moore and Nordgren (cf. [6, 8]), while we require the infinite-dimensional extension of this theory from [2].

2. Some Background Material

For a separable, complex, Hilbert space $\mathscr{F}$ we denote by $L^2(\mathscr{F})$ the set of all Lebesgue measurable, square integrable functions $f\colon \mathbb{T} \to \mathscr{F}$, where $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ is the unit circle in $\mathbb{C}$. The space $H^2(\mathscr{F})$ can be indentified isometrically with the subspace of those functions $f \in L^2(\mathscr{F})$ such that

$$\int_0^{2\pi} e^{int} f(e^{it})\, dt = 0, \qquad n = 1, 2, \ldots .$$

We will make this identification without further ado. Thus, an element $f \in H^2(\mathscr{F})$ is an analytic function on $\mathbb{D} = \{\lambda : |\lambda| < 1\}$, and a measurable function defined on $\mathbb{T}$. Similarly, if $\mathscr{G}$ is another separable Hilbert space, and

$$\phi\colon \mathbb{D} \to \mathscr{L}(\mathscr{G}, \mathscr{F})$$

is a bounded analytic function, then one can define the boundary values $\phi(\zeta)$ for almost every $\zeta \in \mathbb{T}$. We denote by $H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{G}))$ the set of all bounded analytic functions $\phi\colon \mathbb{D} \to \mathscr{L}(\mathscr{F}, \mathscr{G})$. We recall that such a function ϕ is said to be inner [resp. two-sided inner] if $\phi(\zeta)$ is an isometry [resp. a unitary operator] for almost every $\zeta \in \mathbb{T}$. For every $\phi \in H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{G}))$ there is a multiplication operator $M_\phi\colon L^2(\mathscr{F}) \to L^2(\mathscr{G})$ defined by

$$(M_\phi f)(\zeta) = \phi(\zeta)\, f(\zeta), \qquad f \in L^2(\mathscr{F}),$$

for almost every $\zeta \in \mathbb{T}$. Then ϕ is inner [resp. two-sided inner] if and only if M_ϕ is an isometry [resp. a unitary operator]. The function ϕ is said to be outer if $M_\phi H^2(\mathscr{F})$ is dense in $H^2(\mathscr{G})$.

One reason why inner functions are important is the Beurling–Lax–Halmos theorem stated below.

2.1. Theorem. *A subspace $\mathscr{U} \subset H^2(\mathscr{F})$ is invariant for U_+ if and only if there exists a Hilbert space $\mathscr{G}$, and an inner function $\Theta \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ such that $\mathscr{U} = M_\Theta H^2(\mathscr{G})$.*

Given an inner function $\Theta \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$, one can construct the subspace

$$\mathscr{H}(\Theta) = H^2(\mathscr{F}) \ominus M_\Theta H^2(\mathscr{G})$$

invariant for U_+^*, and the operator $S(\Theta)$ on $\mathscr{H}(\Theta)$ defined by

$$S(\Theta) = P_{\mathscr{H}(\Theta)} U_+ \mid \mathscr{H}(\Theta) \qquad \text{or} \qquad S(\Theta)^* = U_+^* \mid \mathscr{H}(\Theta).$$

We will have occasion to use the scalar-valued Hardy spaces H^1, H^2, and H^∞, and the Nevanlinna class N^+, for which we refer to [3]. We have the following characterization of operators of class C_0 from [10].

2.2. THEOREM. *Let $\Theta \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ be an inner function. The following conditions are equivalent*:

(i) *$S(\Theta)$ is an operator of class C_0*;

(ii) *there exists an inner function $\varphi \in H^\infty$ such that $M_\Theta H^2(\mathscr{G}) \supset \varphi H^2(\mathscr{F})$*;

(iii) *there exists an inner function $\varphi \in H^\infty$, and an inner function $\Omega \in H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{G}))$ such that*

$$\Theta(\lambda)\, \Omega(\lambda) = \varphi(\lambda)\, I_{\mathscr{F}}, \qquad \lambda \in \mathbb{D}.$$

The functions φ that satisfy (ii) *also satisfy* (iii). *Moreover, if $S(\Theta)$ is of class C_0 then Θ and Ω are two-sided inner functions and $\Omega(\lambda)\, \Theta(\lambda) = \varphi(\lambda)\, I_{\mathscr{G}}$, $\lambda \in \mathbb{D}$.*

We conclude this section with a statement of the classification theorem for operators of class C_0 from [1].

2.3. THEOREM. *Let Θ be an inner function such that $S(\Theta)$ is an operator of class C_0. There exists a sequence $\{\theta_j : j \geq 0\}$ of inner functions, uniquely determined up to constant factors of absolute value one, such that*

(i) *$\theta_{j+1} \mid \theta_j$ for all j; and*

(ii) *$S(\Theta)$ is quasisimilar to $\bigoplus_{j=0}^\infty S(\theta_j)$.*

The operator $\bigoplus_{j=0}^\infty S(\theta_j)$ is known as the *Jordan model* of the operator $S(\Theta)$.

3. INVARIANT FACTORS

We recall that for a matrix ϕ over H^∞, $\mathscr{D}_j(\phi)$ $(j \geq 1)$ denotes the greatest common inner divisor of all minors of order j of ϕ. Of course, $\mathscr{D}_j(\phi) = 0$ if all such minors are zero. It will be convenient to define $\mathscr{D}_j(\phi)=0$ if j is greater than the number of rows or columns of ϕ (in case ϕ is finite). Clearly $\mathscr{D}_j(\phi)$ divides $\mathscr{D}_{j+1}(\phi)$ for all $j \geqslant 1$, so we can define the *invariant factors* $\mathscr{E}_j(\phi)$ of ϕ as

$$\mathscr{E}_1(\phi)=\mathscr{D}_1(\phi),$$

$$\mathscr{E}_j(\phi)=\mathscr{D}_j(\phi)/\mathscr{D}_{j-1}(\phi) \qquad \text{if} \quad \mathscr{D}_j(\phi)\neq 0 \text{ and } j \geqslant 2,$$

$$\mathscr{E}_j(\phi)=0 \qquad \text{if} \quad \mathscr{D}_j(\phi)=0.$$

The following theorem was proved in [9].

3.1. THEOREM. *For every matrix ϕ, $\mathscr{E}_j(\phi)$ divides $\mathscr{E}_{j+1}(\phi)$, $j \geqslant 1$.*

The following result was noted in [7]; we show here that it is a consequence of the results in [2].

3.2. THEOREM. *Let $\Theta \in H^\infty(\mathscr{L}(\mathscr{G},\mathscr{F}))$, $\Omega \in H^\infty(\mathscr{L}(\mathscr{F},\mathscr{G}))$, and $\varphi \in H^\infty$ be inner functions such that $\Theta(\lambda)\,\Omega(\lambda)=\varphi(\lambda)\,I_{\mathscr{F}}$, $\lambda \in \mathbb{D}$. Then the Jordan model $\bigoplus_{j=0}^\infty S(\theta_j)$ of $S(\Theta)$ can be calculated as follows*

$$\theta_j=\varphi/\mathscr{E}_{j+1}(\Omega), \qquad j<\dim(\mathscr{F})$$

$$\theta_j=1, \qquad j \geqslant \dim(\mathscr{F}).$$

Proof. It is well known that $\theta_j=1$ for $j \geqslant \dim(\mathscr{F})$, so we concentrate on $j<\dim(\mathscr{F})$. By Corollary 3.3 of [2], $\theta_0\theta_1\cdots\theta_{j-1}$ is the least scalar multiple of the exterior power $\Theta(\lambda)^{\wedge j}$. The relation

$$\Theta(\lambda)^{\wedge j}\Omega(\lambda)^{\wedge j}=\varphi(\lambda)^j I_{\mathscr{F}^{\wedge j}}$$

shows that $\theta_0\theta_1, ..., \theta_{j-1}$ must equal φ^j divided by the greatest inner divisor of all the entries of $\Omega(\lambda)^{\wedge j}$. Since this inner divisor is exactly $\mathscr{D}_j(\Omega)$, we deduce

$$\theta_0\theta_1\cdots\theta_{j-1}=\varphi^j/\mathscr{D}_j(\Omega).$$

Analogously,

$$\theta_0\theta_1\cdots\theta_j=\varphi^{j+1}/\mathscr{D}_{j+1}(\Omega),$$

and dividing the last two relations we obtain the desired result.

We will need an extension of the definition of invariant factors.

3.3. DEFINITION. Let $\mathscr{X} \subset H^2(\mathscr{F})$ be an arbitrary set, and $1 \leqslant j \leqslant \dim(\mathscr{F})$ an integer. Let $\{e_1, e_2, ..., e_j\}$ be an orthonormal system in $\mathscr{F}$, and let $x_1, x_2, ..., x_j \in \mathscr{X}$. The function

$$\det[(x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j}, \qquad \lambda \in \mathbb{D},$$

is called a *minor of order j of $\mathscr{X}$*. The function $\mathscr{D}_j(\mathscr{X})$ is the greatest common inner divisor of all minors of order j of $\mathscr{X}$. The *invariant factors* $\mathscr{E}_j(\mathscr{X})$ are defined as

$$\mathscr{E}_1(\mathscr{X}) = \mathscr{D}_1(\mathscr{X}),$$

$$\mathscr{E}_k(\mathscr{X}) = \mathscr{D}_k(\mathscr{X})/\mathscr{D}_{k-1}(\mathscr{X}), \qquad 2 \leqslant k \leqslant \dim(\mathscr{F}),\ \mathscr{D}_k(\mathscr{X}) \neq 0,$$

and

$$\mathscr{E}_k(\mathscr{X}) = 0, \qquad 2 \leqslant k \leqslant \dim(\mathscr{F}),\ \mathscr{D}_k(\mathscr{X}) = 0.$$

For $j > \dim(\mathscr{F})$ we set $\mathscr{D}_j(\mathscr{X}) = \mathscr{E}_j(\mathscr{X}) = 0$.

We should emphasize here that a minor of order j of $\mathscr{X}$ is not generally an element of H^p for some $p > 0$. These determinants belong, however, the Nevanlinna class N^+, so they can be factored as an inner function times an outer function. Thus the definition of $\mathscr{D}_j(\mathscr{X})$ makes sense since the outer factors are irrelevant.

3.4. THEOREM. *Let $\mathscr{X} \subset H^2(\mathscr{F})$ be an arbitrary set, and let $\phi \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ be an inner function such that*

$$\bigvee_{n=0}^{\infty} U_+^n \mathscr{X} = M_\phi H^2(\mathscr{G}).$$

Then we have $\mathscr{D}_j(\mathscr{X}) = \mathscr{D}_j(\phi)$ and $\mathscr{E}_j(\mathscr{X}) = \mathscr{E}_j(\phi)$ for all $j \geqslant 1$.

Proof. It suffices to show that $\mathscr{D}_j(\mathscr{X}) = \mathscr{D}_j(\phi)$ for $j \leqslant \dim(\mathscr{F})$. Let therefore $j \leqslant \dim(\mathscr{F})$ be fixed. Denote by $\mathscr{Y}$ the linear manifold generated by $\bigcup_{n=0}^{\infty} U_+^n \mathscr{X}$, and let $\mathscr{Z} = \mathscr{Y}^-$, so that

$$\mathscr{Z} = M_\phi H^2(\mathscr{G}).$$

We claim that it suffices to prove that

$$\mathscr{D}_j(\mathscr{X}) = \mathscr{D}_j(\mathscr{Z}). \tag{3.5}$$

Indeed, assume this equality has been proved for all subsets $\mathscr{X} \subset H^2(\mathscr{F})$. Fix an orthonormal basis $\{e_i\}_i$ of $\mathscr{G}$, and set

$$\mathscr{X}' = \{M_\phi e_i\}_i.$$

It is clear that

$$\begin{aligned} \bigvee_{n=0}^{\infty} U_+^n \mathscr{X}' &= \bigvee_{n=0}^{\infty} \bigvee_i M_\phi U_+^n e_i \\ &= M_\phi H^2(\mathscr{G}) \\ &= \mathscr{Z}, \end{aligned}$$

so that we would have from (3.5) that

$$\mathscr{D}_j(\mathscr{X}) = \mathscr{D}_j(\mathscr{X}') = \mathscr{D}_j(\mathscr{Z}).$$

But it is clear that $\mathscr{D}_j(\mathscr{X}') = \mathscr{D}_j(\phi)$ because the elements $M_\phi e_i$ are exactly the "columns" of the matrix ϕ. We set out therefore to prove (3.5). We first note that the inclusions $\mathscr{X} \subset \mathscr{Y} \subset \mathscr{Z}$ imply that

$$\mathscr{D}_j(\mathscr{Z}) \,|\mathscr{D}_j(\mathscr{Y})|\, \mathscr{D}_j(\mathscr{X}), \tag{3.6}$$

where the vertical bar means "divides." Assume on the other hand that $y_1, y_2, ..., y_j \in \mathscr{Y}$ so that each y_k is a linear combination of vectors of the form $U_+^n x$ with $n \geqslant 0$ and $x \in \mathscr{X}$. Assume further that $\{e_1, e_2, ..., e_j\}$ is an orthonormal system in $\mathscr{F}$. The multilinearity of the determinant implies that the minor $\det[(y_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j}$ is a linear combination of functions of the form

$$\det[(\lambda^{n_i} x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j} = \lambda^n \det[(x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j},$$

where $n = n_1 + n_2 + \cdots + n_j$ and $x_i \in \mathscr{X}$. It follows therefore that $\mathscr{D}_j(\mathscr{X})$ divides every minor of order j of $\mathscr{Y}$ and hence $\mathscr{D}_j(\mathscr{X}) \,|\, \mathscr{D}_j(\mathscr{Y})$. By (3.6) we have $\mathscr{D}_j(\mathscr{X}) = \mathscr{D}_j(\mathscr{Y})$, so it remains to prove that $\mathscr{D}_j(\mathscr{Y}) = \mathscr{D}_j(\mathscr{Z})$. Assume therefore that $z_1, z_2, ..., z_j \in \mathscr{Z}$, and $\{e_1, e_2, ..., e_j\}$ is an orthonormal system in $\mathscr{F}$. For each i, $1 \leqslant i \leqslant j$, choose a sequence $y_i^{(n)} \in \mathscr{Y}$ such that $\lim_{n \to \infty} \|z_i - y_i^{(n)}\| = 0$. Next, choose an outer function $\psi \in N^+$ such that all the functions $(\psi(\lambda)\, y_i^{(n)}(\lambda), e_k)$ and $(\psi(\lambda)\, z_i(\lambda), e_k)$ are bounded (the existence of such a function ψ is left as an exercise to the reader). In order to simplify notation for the remainder of this proof, a determinant of the form $\det[(x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j}$ will be denoted $\Delta(x_1, x_2, ..., x_j)$. Since all the entries in the determinants $\Delta(\psi y_1^{(n_1)}, \psi y_2^{(n_2)}, ..., \psi y_j^{(n_j)})$ are bounded, we deduce that $\lim_{n_1 \to \infty} \Delta(\psi y_1^{(n_1)}, \psi y_2^{(n_2)}, ..., \psi y_j^{(n_j)}) = \Delta(\psi z_1, \psi y_2^{(n_2)}, ..., \psi y_j^{(n_j)})$ in the H^2 norm. Therefore $\mathscr{D}_j(\mathscr{Y})$ divides the limit function $\Delta(\psi z_1, \psi y_2^{(n_2)}, ..., \psi y_j^{(n_j)})$. We proceed analogously, letting $n_2, n_3, ..., n_j$ tend to infinity one at a time and conclude that $\mathscr{D}_j(\mathscr{Y})$ divides

$$\begin{aligned}\Delta(\psi z_1, \psi z_2, ..., \psi z_j) &= \psi^j \Delta(z_1, z_2, ..., z_j) \\ &= \psi^j \det[(z_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j}.\end{aligned}$$

Since ψ is outer, we conclude that $\mathscr{D}_j(\mathscr{Y})$ divides every minor of order j of $\mathscr{Z}$, whence $\mathscr{D}_j(\mathscr{Y}) \,|\, \mathscr{D}_j(\mathscr{Z})$. By (3.6), $\mathscr{D}_j(\mathscr{Y}) = \mathscr{D}_j(\mathscr{Z})$, and this concludes the proof.

In the following result we use the symbol "$\wedge$" to denote the greatest common inner divisor.

3.7. PROPOSITION. *Let* $\mathscr{X} \subset H^2(\mathscr{F})$ *be a subset,* $\varphi \in H^\infty$ *an inner function, and* $j \leqslant \dim(\mathscr{F})$. *We have*

$$\mathscr{D}_j(\mathscr{X} \cup \varphi H^2(\mathscr{F})) = \bigwedge_{p=0}^{j} \varphi^p \mathscr{D}_{j-p}(\mathscr{X}).$$

Proof. It is easy to see that

$$\mathscr{D}_j(\mathscr{X} \cup \varphi H^2(\mathscr{F})) | \varphi^p \mathscr{D}_{j-p}(\mathscr{X}). \tag{3.8}$$

Indeed, let $\det[(x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j-p}$ be a minor of order $j-p$ of $\mathscr{X}$, and let $\{e_{j-p+1}, ..., e_j\}$ be such that $\{e_1, e_2, ..., e_j\}$ is an orthonormal set. If we set $x_i = \varphi e_i$, $j-p < i \leqslant j$, then

$$\det[(x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j} = \varphi^p \det[(x_i(\lambda), e_k)]_{1 \leqslant i,k \leqslant j-p}.$$

Since the left-hand side is a minor of order j of $\mathscr{X} \cup \varphi H^2(\mathscr{F})$, it follows that $\mathscr{D}_j(\mathscr{X} \cup \varphi H^2(\mathscr{F}))$ divides the right-hand side. This observation implies immediately (3.8). On the other hand, any minor of order j of $\mathscr{X} \cup \varphi H^2(\mathscr{F})$ is formed with some, say p, vectors from $H^2(\mathscr{F})$, and $j-p$ vectors from $\mathscr{X}$. It follows that such a minor is a sum of factors of the form AB, with A a minor of order p of $\varphi H^2(\mathscr{F})$, and B a minor of order $j-p$ of $\mathscr{X}$. Therefore we deduce that

$$\bigwedge_{p=0}^{j} \varphi^p \mathscr{D}_{j-p}(\mathscr{X}) = \bigwedge_{p=0}^{j} \mathscr{D}_p(\varphi H^2(\mathscr{F}))\, \mathscr{D}_{j-p}(\mathscr{X})$$

must divide $\mathscr{D}_p(\mathscr{X} \cup \varphi H^2(\mathscr{F}))$. The proposition is proved.

4. JORDAN MODELS

We are now ready to treat the problem stated in the Introduction. Throughout this section $\mathscr{M}$ is a fixed subset of $H^2(\mathscr{F})$, and $\Theta \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ is an inner function such that

$$\mathscr{H}(\Theta) = \bigvee_{n=0}^{\infty} U_+^{*n} \mathscr{M}.$$

4.1. LEMMA. *The operator* $S(\Theta)$ *is of class* C_0 *if and only if there exists an inner function* $\varphi \in H^\infty$ *such that*

$$\bar{\varphi} \mathscr{M} \subset L^2(\mathscr{F}) \ominus H^2(\mathscr{F}). \tag{4.2}$$

Proof. By Theorem 2.2, $S(\Theta)$ is of class C_0 if and only if $M_\Theta H^2(\mathscr{G}) \supset \varphi H^2(\mathscr{F})$ for some inner function $\varphi \in H^\infty$ or, equivalently, if

$$\mathscr{H}(\Theta) \subset H^2(\mathscr{F}) \ominus \varphi H^2(\mathscr{F}).$$

Since $\mathcal{M}$ is, by definition, a cyclic set for $U_+^* \mid \mathcal{H}(\Theta)$, and since $H^2(\mathcal{F}) \ominus \varphi H^2(\mathcal{F})$ is invariant for U_+^*, this latter inclusion happens if and only if

$$\mathcal{M} \subset H^2(\mathcal{F}) \ominus \varphi H^2(\mathcal{F}).$$

Since $\mathcal{M} \subset H^2(\mathcal{F})$, this inclusion is equivalent with

$$\mathcal{M} \subset L^2(\mathcal{F}) \ominus \varphi H^2(\mathcal{F}),$$

and hence with

$$\bar{\varphi}\mathcal{M} \subset \bar{\varphi} L^2(\mathcal{F}) \ominus \bar{\varphi}\varphi H^2(\mathcal{F}) = L^2(\mathcal{F}) \ominus H^2(\mathcal{F}).$$

The lemma follows.

We denote by U the bilateral shift on $L^2(\mathcal{F})$, which is the unitary extension of U_+.

4.3. LEMMA. *For every inner function φ, the space $L^2(\mathcal{F}) \ominus \bar{\varphi} M_\Theta H^2(\mathcal{G})$ is the invariant subspace for U^* generated by $\bar{\varphi}\mathcal{M}$ and $L^2(\mathcal{F}) \ominus \bar{\varphi} H^2(\mathcal{F})$.*

Proof. Since multiplication by $\bar{\varphi}$ is a unitary operator on $L^2(\mathcal{F})$ commuting with U^*, it suffices to show that $L^2(\mathcal{F}) \ominus M_\Theta H^2(\mathcal{G})$ is the invariant subspace for U^* generated by $\mathcal{M}$ and $L^2(\mathcal{F}) \ominus H^2(\mathcal{F})$. This statement is then equivalent to

$$L^2(\mathcal{F}) \ominus M_\Theta H^2(\mathcal{G}) = (L^2(\mathcal{F}) \ominus H^2(\mathcal{F})) \vee \left(\bigvee_{n=0}^{\infty} U^{*n}\mathcal{M} \right)$$

and, since both sides contain $L^2(\mathcal{F}) \ominus H^2(\mathcal{F})$, this equality is equivalent to

$$H^2(\mathcal{F}) \ominus M_\Theta H^2(\mathcal{G}) = \bigvee_{n=0}^{\infty} P_{H^2(\mathcal{F})} U^{*n}\mathcal{M}.$$

This last equality is, however, obvious because

$$H^2(\mathcal{F}) \ominus M_\Theta H^2(\mathcal{G}) = \mathcal{H}(\Theta) \qquad \text{and} \qquad P_{H^2(\mathcal{F})} U^{*n} \mid H^2(\mathcal{F}) = U_+^{*n}.$$

4.4. LEMMA. *Assume that $\Omega \in H^\infty(\mathcal{L}(\mathcal{F}, \mathcal{G}))$ and $\varphi \in H^\infty$ are inner functions such that $\Theta(\lambda)\,\Omega(\lambda) = \varphi(\lambda)\, I_{\mathcal{F}}$, $\lambda \in \mathbb{D}$. Then*

$$\bar{\varphi} M_\Theta H^2(\mathcal{G}) = M_\Omega^* H^2(\mathcal{G}).$$

Proof. This is an immediate consequence of the equations

$$\bar{\varphi}(\zeta)\,\Theta(\zeta) = (\varphi(\zeta)\, I_{\mathcal{F}})^*\,\Theta(\zeta) = \Omega(\zeta)^*\,\Theta(\zeta)^*\,\Theta(\zeta) = \Omega(\zeta)^*,$$

which hold for almost every $\zeta \in \mathbb{T}$ because φ and Θ are inner.

In order to put together the preceding observations, we need the unitary operator $R\colon L^2(\mathscr{F}) \to L^2(\mathscr{F})$ defined by

$$(Rf)(\zeta) = \bar{\zeta} f(\bar{\zeta}), \qquad \zeta \in \mathbb{T}, \quad f \in L^2(\mathscr{F}).$$

The following properties of R are easily verified:

$$R = R^* = R^{-1},$$

$$UR = RU^*,$$

$$R(H^2(\mathscr{F}) \ominus L^2(\mathscr{F})) = H^2(\mathscr{F}).$$

Of course, there is a corresponding operator R on $L^2(\mathscr{G})$. If $\phi \in H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{G}))$, we have

$$RM_\phi^* = M_{\phi^\sim} R,$$

where $\phi^\sim \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ is defined by

$$\phi^\sim(\lambda) = \phi(\bar{\lambda})^*, \qquad \lambda \in \mathbb{D}.$$

4.5. PROPOSITION. *Let* $\varphi \in H^\infty$ *be an inner function such that* $\bar{\varphi}\mathscr{M} \subset L^2(\mathscr{F}) \ominus H^2(\mathscr{F})$, *and let* $\Omega \in H^\infty(\mathscr{L}(\mathscr{F}, \mathscr{G}))$ *be an inner function such that*

$$\Theta(\lambda)\,\Omega(\lambda) = \varphi(\lambda)\, I_{\mathscr{F}}, \qquad \lambda \in \mathbb{D}.$$

Then $M_{\Omega^\sim} H^2(\mathscr{G})$ *is the invariant subspace for* U_+ *generated by* $R(\bar{\varphi}\mathscr{M})$ *and* $\varphi^\sim H^2(\mathscr{F})$.

Proof. Since $\bar{\varphi}\mathscr{M} \subset L^2(\mathscr{F}) \ominus H^2(\mathscr{F})$, we have $R(\bar{\varphi}\mathscr{M}) \subset H^2(\mathscr{F})$, so the statement makes sense. Next, let us note that

$$\begin{aligned} R(L^2(\mathscr{F}) \ominus \bar{\varphi} M_\Theta H^2(\mathscr{G})) &= R(L^2(\mathscr{F}) \ominus M_\Omega^* H^2(\mathscr{G})) \\ &= RM_\Omega^*(L^2(\mathscr{G}) \ominus H^2(\mathscr{G})) \\ &= M_{\Omega^\sim} R(L^2(\mathscr{G}) \ominus H^2(\mathscr{G})) \\ &= M_{\Omega^\sim} H^2(\mathscr{G}), \end{aligned}$$

and analogously,

$$R(L^2(\mathscr{F}) \ominus \bar{\varphi} H^2(\mathscr{F})) = \varphi^\sim H^2(\mathscr{F}).$$

Since $L^2(\mathscr{F}) \ominus \bar{\varphi} M_\Theta H^2(\mathscr{G})$ is the invariant subspace for U^* generated by $\bar{\varphi}\mathscr{M}$ and $L^2(\mathscr{F}) \ominus \bar{\varphi} H^2(\mathscr{F})$, it follows at once that

$$M_{\Omega^\sim} H^2(\mathscr{G}) = R(L^2(\mathscr{F}) \ominus \bar{\varphi} M_\Theta H^2(\mathscr{G}))$$

is the invariant subspace for U generated by $R(\bar{\varphi}\mathcal{M})$ and $\varphi^{\sim}H^2(\mathcal{F})$. The proposition is proved.

Now, clearly $\mathcal{E}_j(\Omega^{\sim}) = \mathcal{E}_j(\Omega)^{\sim}$, so that the Jordan model of $S(\Theta)$ is easily calculated by Theorems 3.4 and 3.2. The following result puts the resulting formulas in a more elegant form, thus generalizing Frazho's result from [4].

4.6. THEOREM. *Let $\mathcal{M} \subset H^2(\mathcal{F})$ be an arbitrary subset, and $\varphi \in H^\infty$ an inner function such that $\bar{\varphi}\mathcal{M} \subset L^2(\mathcal{F}) \ominus H^2(\mathcal{F})$. Then the operator $T = P_{\mathcal{N}} U_+ \mid \mathcal{N}$, where $\mathcal{N} = \bigvee_{n=0}^{\infty} U_+^{*n}\mathcal{M}$, is an operator of class C_0, and its Jordan model $\oplus_{j=0}^{\infty} S(\theta_j)$ can be calculated as follows*

$$\theta_j = \varphi/\mathcal{E}_{j+1}(R(\bar{\varphi}\mathcal{M}))^{\sim} \wedge \varphi \qquad \text{if} \quad j < \dim(\mathcal{F}),$$

$$\theta_j = 1 \qquad \text{if} \quad j \geqslant \dim(\mathcal{F}).$$

Proof. Let Θ and Ω be as in the preceding results, and let $j < \dim(\mathcal{F})$. Then Theorem 3.2 gives

$$\theta_j = \varphi/\mathcal{E}_{j+1}(\Omega).$$

Now, Theorem 3.4 shows that

$$\begin{aligned} \mathcal{E}_{j+1}(\Omega)^{\sim} &= \mathcal{E}_{j+1}(\Omega^{\sim}) \\ &= \mathcal{E}_{j+1}(R(\bar{\varphi}\mathcal{M}) \cup \varphi^{\sim}H^2(\mathcal{F})). \end{aligned}$$

Thus the theorem will be proved if we can show that

$$\mathcal{E}_j(R(\bar{\varphi}\mathcal{M}) \cup \varphi^{\sim}H^2(\mathcal{F})) = \mathcal{E}_j(R(\bar{\varphi}\mathcal{M})) \wedge \varphi^{\sim}$$

for all $j \leqslant \dim \mathcal{F}$. This equality in turn is equivalent to

$$\begin{aligned} &\mathcal{D}_j(R(\bar{\varphi}\mathcal{M}) \cup \varphi^{\sim}H^2(\mathcal{F})) \\ &\quad = (\mathcal{E}_1(R(\bar{\varphi}\mathcal{M})) \wedge \varphi^{\sim})(\mathcal{E}_2(R(\bar{\varphi}\mathcal{M}) \wedge \varphi^{\sim}) \cdots (\mathcal{E}_j(R(\bar{\varphi}\mathcal{M})) \wedge \varphi^{\sim}) \end{aligned}$$

for all $j \leqslant \dim \mathcal{F}$. In order to simplify notation, we set $\varepsilon_j = \mathcal{E}_j(R(\bar{\varphi}\mathcal{M}))$, $\delta_j = \mathcal{D}_j(R(\bar{\varphi}\mathcal{M}))$, and $d_j = \mathcal{D}_j(R(\bar{\varphi}\mathcal{M}) \cup \varphi^{\sim}H^2(\mathcal{F}))$ so the relation to be proved is

$$d_j = (\varepsilon_1 \wedge \varphi^{\sim})(\varepsilon_2 \wedge \varphi^{\sim}) \cdots (\varepsilon_j \wedge \varphi^{\sim}). \tag{4.7}$$

By Proposition 3.7 we have

$$d_j = \bigwedge_{p=0}^{j} \varphi^{\sim p}\,\delta_{j-p}, \tag{4.8}$$

and this will allow us to prove (4.7) by induction on j. For $j=1$ (4.7) coincides with (4.8). Assume that (4.7) is verified with j replaced by $j-1$, and rewrite (4.8) as

$$\begin{aligned} d_j &= \bigwedge_{p=0}^{j} \varphi^{\sim p}\,\delta_{j-p} \\ &= \delta_j \wedge \left(\bigwedge_{p=1}^{j} \varphi^{\sim p}\,\delta_{j-p} \right) \\ &= \delta_j \wedge \varphi^{\sim} \left(\bigwedge_{p=1}^{j} \varphi^{\sim p-1}\,\delta_{j-p} \right) \\ &= \delta_j \wedge \varphi^{\sim} \left(\bigwedge_{q=0}^{j-1} \varphi^{\sim q}\,\delta_{(j-1)-q} \right) \\ &= \delta_j \wedge \varphi^{\sim} d_{j-1}, \end{aligned}$$

where we also used (4.8) with j replaced by $j-1$. Apply now the induction hypothesis to get

$$\begin{aligned} d_j &= \delta_j \wedge [\varphi^{\sim}(\varepsilon_1 \wedge \varphi^{\sim})(\varepsilon_2 \wedge \varphi^{\sim}) \cdots (\varepsilon_{j-1} \wedge \varphi^{\sim})] \\ &= (\varepsilon_1 \varepsilon_2 \cdots \varepsilon_j) \wedge [\varphi^{\sim}(\varepsilon_1 \wedge \varphi^{\sim})(\varepsilon_2 \wedge \varphi^{\sim}) \cdots (\varepsilon_{j-1} \wedge \varphi^{\sim})] \\ &= (\varepsilon_1 \wedge \varphi^{\sim})(\varepsilon_2 \wedge \varphi^{\sim}) \cdots (\varepsilon_{j-1} \wedge \varphi^{\sim}) \\ &\quad \times \left[\left(\frac{\varepsilon_1}{\varepsilon_1 \wedge \varphi^{\sim}} \frac{\varepsilon_2}{\varepsilon_2 \wedge \varphi^{\sim}} \cdots \frac{\varepsilon_{j-1}}{\varepsilon_{j-1} \wedge \varphi^{\sim}} \varepsilon_j \right) \wedge \varphi^{\sim} \right] \\ &= (\varepsilon_1 \wedge \varphi^{\sim})(\varepsilon_2 \wedge \varphi^{\sim}) \cdots (\varepsilon_j \wedge \varphi^{\sim}) \\ &\quad \times \left[\left(\frac{\varepsilon_1}{\varepsilon_1 \wedge \varphi^{\sim}} \frac{\varepsilon_2}{\varepsilon_2 \wedge \varphi^{\sim}} \cdots \frac{\varepsilon_{j-1}}{\varepsilon_j \wedge \varphi^{\sim}} \right) \wedge \left(\frac{\varphi^{\sim}}{\varepsilon_j \wedge \varphi^{\sim}} \right) \right]. \end{aligned}$$

To conclude the proof of (4.7) it suffices to show that $\varphi^{\sim}/\varepsilon_j \wedge \varphi^{\sim}$ is relatively prime with $\varepsilon_i/\varepsilon_i \wedge \varphi^{\sim}$ for $i \leqslant j$. Now, we know from Theorem 3.1 that $\varepsilon_i | \varepsilon_j$ for $i \leqslant j$, and hence $\varphi^{\sim}/\varepsilon_j \wedge \varphi^{\sim} \,|\, \varphi^{\sim}/\varepsilon_i \wedge \varphi^{\sim}$ for $i \leqslant j$. It suffices therefore to prove that $\varphi^{\sim}/\varepsilon_i \wedge \varphi^{\sim}$ is relatively prime with $\varepsilon_i/\varepsilon_i \wedge \varphi^{\sim}$, and this is obvious. The proof of (4.7) by induction, and that of the theorem, is complete.

5. Outer Factors

In this section we apply the previous results to an important class of subsets $\mathcal{M} \subset H^2(\mathcal{F})$. These sets arise in the following manner. Assume that

$N\colon \mathbb{T} \to \mathscr{L}(\mathscr{G})$ is a weakly measurable, positive definite, essentially bounded function. Here, weak measurability means that the functions $\zeta \to (N(\zeta)f, g)$, $\zeta \in \mathbb{T}$, are Lebesgue measurable for all f, $g \in \mathscr{G}$. We assume that N is factorable; that is, there exists a function $\phi \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ such that $N(\zeta) = \phi(\zeta)^*\phi(\zeta)$ for almost every $\zeta \in \mathbb{T}$. The function ϕ is not uniquely determined, but if we assume it to be outer it is uniquely determined up to a constant unitary factor; if ϕ is outer, then it is called an outer factor of N. We can now define

$$\mathscr{M} = \{M_\phi g : g \in \mathscr{G}\} \subset H^2(\mathscr{F}),$$

the set of "columns" of ϕ. As before, we choose an inner function $\Theta \in H^\infty(\mathscr{L}(\mathscr{K}, \mathscr{F}))$ such that

$$\mathscr{H}(\Theta) = \bigvee_{n=0}^{\infty} U_+^* \mathscr{M}.$$

The problem now is to determine whether $S(\Theta)$ is an operator of class C_0, and to calculate its Jordan model without actually finding the outer factor ϕ or the characteristic function Θ. We will need to establish an easy lemma before stating a solution to this problem. Throughout this section, $\mathscr{G}$, $\mathscr{F}$, $\mathscr{K}$, N, ϕ, and Θ will be fixed with the above properties, and ϕ will be assumed to be outer.

5.1. Lemma. *Let $\phi \in H^\infty(\mathscr{L}(\mathscr{G}, \mathscr{F}))$ be an outer function, and $\mathscr{X} \subset H^2(\mathscr{F})$. Then we have $\mathscr{E}_j(\mathscr{X}) = \mathscr{E}_j(M_{\phi^\sim}\mathscr{X})$, $j \geq 1$.*

Proof. Fix an inner function $\Omega \in H^\infty(\mathscr{L}(\mathscr{K}, \mathscr{F}))$ such that

$$\bigvee_{n=0}^{\infty} U_+^n \mathscr{X} = M_\Omega H^2(\mathscr{K}).$$

By Theorem 3.4 we have

$$\begin{aligned}
\mathscr{E}_j(M_{\phi^\sim}\mathscr{X}) &= \mathscr{E}_j\left(\bigvee_{n=0}^{\infty} U_+^n M_{\phi^\sim}\mathscr{X}\right)\\
&= \mathscr{E}_j\left(\bigvee_{n=0}^{\infty} M_{\phi^\sim} U_+^n \mathscr{X}\right)\\
&= \mathscr{E}_j((M_{\phi^\sim} M_\Omega H^2(\mathscr{K}))^-)\\
&= \mathscr{E}_j(\phi^\sim \Omega).
\end{aligned}$$

On the other hand, since ϕ is outer,

$$\begin{aligned}\mathscr{E}_j(\phi^{\sim}\Omega)^{\sim} &= \mathscr{E}_j(\Omega^{\sim}\phi)\\ &= \mathscr{E}_j((M_{\Omega^{\sim}\phi}H^2(\mathscr{G})))^{-}\\ &= \mathscr{E}_j[(M_{\Omega^{\sim}}(M_\phi H^2(\mathscr{G}))^{-})^{-}]\\ &= \mathscr{E}_j((M_{\Omega^{\sim}}H^2(\mathscr{F}))^{-})\\ &= \mathscr{E}_j(\Omega^{\sim})\\ &= \mathscr{E}_j(\Omega)^{\sim}\\ &= \mathscr{E}_j(\mathscr{X})^{\sim}.\end{aligned}$$

Thus,

$$\mathscr{E}_j(M_{\phi^{\sim}}\mathscr{X}) = \mathscr{E}_j(\phi^{\sim}\Omega) = \mathscr{E}_j(\mathscr{X}),$$

as desired.

5.2. THEOREM. *The operator $S(\Theta)$ is of class C_0 if and only if there exists an inner function $\varphi \in H^\infty$ such that*

$$(M_{\bar{\varphi}N}\, g)(\zeta) = \overline{\varphi(\zeta)}\, N(\zeta)\, g, \qquad \zeta \in \mathbb{T},$$

belongs to $L^2(\mathscr{G}) \ominus H^2(\mathscr{G})$ for every $g \in \mathscr{G}$. If φ satisfies this condition, the Jordan model $\bigoplus_{j=0}^{\infty} S(\theta_j)$ of $S(\Theta)$ can be calculated as follows:

$$\theta_j = \varphi/\mathscr{E}_{j+1}(RM_{\bar{\varphi}N}\mathscr{G})^{\sim} \wedge \varphi, \qquad j \geqslant 0.$$

Proof. For $g \in \mathscr{G}$ and $u \in H^2(\mathscr{G})$ we have

$$(M_{\bar{\varphi}N}\, g, u) = (M_{\bar{\varphi}\phi}\, g, M_\phi u).$$

Since ϕ is outer, the functions $M_\phi u$ are dense in $H^2(\mathscr{F})$. We deduce that $M_{\bar{\varphi}N}\, g$ is in $L^2(\mathscr{G}) \ominus H^2(\mathscr{G})$ if and only if $M_{\bar{\varphi}\phi}\, g = \bar{\varphi} M_\phi\, g$ is in $L^2(\mathscr{F}) \ominus H^2(\mathscr{F})$. The first statement of the theorem follows immediately from Lemma 4.1. The formulas

$$\theta_j = \varphi/\mathscr{E}_{j+1}(RM_{\bar{\varphi}\phi}\mathscr{G})^{\sim} \wedge \varphi, \qquad j \geqslant 0$$

follow from Theorem 4.6, so it remains to prove that

$$\mathscr{E}_j(RM_{\bar{\varphi}\phi}\mathscr{G}) = \mathscr{E}_j(RM_{\bar{\varphi}N}\mathscr{G}), \qquad j \geqslant 1.$$

This, however, follows immediately from Lemma 5.1 because

$$RM_{\bar{\varphi}N} = RM_\phi^* M_{\bar{\varphi}\phi} = M_{\phi^{\sim}} RM_{\bar{\varphi}\phi}.$$

The theorem is proved.

References

1. H. Bercovici, C. Foias, and B. Sz-Nagy, Compléments à l'étude des opérateurs de classe C_0, III, *Acta Sci. Math. (Szeged)* **37** (1975), 313–322.
2. H. Bercovici and D. Voiculescu, Tensor operations on characteristic functions of C_0 contractions, *Acta Sci. Math. (Szeged)* **39** (1977), 205–231.
3. P. L. Duren, "H^p Spaces," Academic Press, New York, 1970.
4. A. E. Frazho, Infinite-dimensional Jordan models and Smith McMillan forms, *Integral Equations Operator Theory* **5** (1982), 184–192.
5. A. E. Frazho, Infinite-dimensional Jordan models and Smith McMillan forms, II, *Acta Sci. Math. (Szeged)* **46** (1983), 317–321.
6. B. Moore, III and E. A. Nordgren, On quasiequivalence and quasisimilarity, *Acta Sci. Math. (Szeged)* **34** (1973), 311–316.
7. V. Müller, On Jordan models of C_0-contractions, *Acta Sci. Math. (Szeged)* **40** (1978), 309–313.
8. E. A. Nordgren, On quasiequivalence of matrices over H^∞, *Acta Sci. Math. (Szeged)* **34** (1973), 301–310.
9. B. Sz-Nagy, Diagonalization of matrices over H^∞, *Acta Sci. Math. (Szeged)* **38** (1976), 223–238.
10. B. Sz.-Nagy and C. Foias, "Harmonic Analysis of Operators on Hilbert Space," North-Holland, Amsterdam, 1970.